THE DIFFEOMORPHISM TYPE
OF CERTAIN $S^3$-BUNDLES OVER $S^4$

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ABSTRACT. In this note we show that the unit tangent bundle of $S^4$ is diffeo-

morphic to the total space of a certain principal $S^3$-bundle over $S^4$, solving a

problem of James and Whitehead.

For more than 50 years it has been known that the $S^3$-bundles over $S^4$ are

classified by $\mathbb{Z} \oplus \mathbb{Z}$. The bundle that corresponds to $(m,n) \in \mathbb{Z} \oplus \mathbb{Z}$ is obtained by

gluing two copies of $\mathbb{R}^4 \times S^3$ together via the diffeomorphism $g_{m,n} : (\mathbb{R}^4 \setminus \{0\}) \times

S^3 \rightarrow (\mathbb{R}^4 \setminus \{0\}) \times S^3$ given by

$$g_{m,n}(u,v) \rightarrow \left( \frac{u}{|u|^2}, \frac{u^m v u^n}{|u|^{n+m}} \right),$$

where we have identified $\mathbb{R}^4$ with $\mathbb{H}$ and $S^3$ with $\{v \in \mathbb{H} \mid |v| = 1\}$ ([Hat], [Steen]).

We will call the bundle obtained from $g_{m,n}$ "the bundle of type $(m,n)$", and we

will denote it by $E_{m,n}$.

The problem of classifying the total spaces of these bundles up to homotopy,

homeomorphism, and diffeomorphism type is still open. It has led to a revolution

in topology that began with Milnor's discovery that most of the bundles of type

$(m, -m + 1)$ are exotic spheres [Mil].

Further motivation for this problem is provided by the many interesting metrics

discovered on these spaces in [GromMey], [GrovZil], [PetWill], [Wil1], and [Wil2].

In 1953 James and Whitehead gave the homotopy classification for

$$(\text{total space, fiber})$$

pairs, except that among the bundles whose third homology group is $\mathbb{Z}/2\mathbb{Z}$ they

were only able to assert that there are at most 2 homotopy types. We will complete

this classification here by proving

Theorem 1. The total spaces of $E_{1,1}$ and $E_{2,0}$ are diffeomorphic, via a diffeo-

morphism that takes a fiber of $E_{1,1}$ to a fiber of $E_{2,0}$.

The complete classification of these total spaces is given independently in

[GromEsc], where it will be shown that there is no orientation preserving homotopy

equivalence between these two total spaces.

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In 1965 Sasson gave the homotopy classification of a family of CW-complexes that represents all of the homotopy types of the total spaces of $S^3$-bundles over $S^4$. It is probably possible to obtain the actual homotopy classification of the total spaces from Sasson's work, although Sasson did not do the computation.

A proof of Theorem 1 was given implicitly in [PetWil], and had been known to us for three or four years. At the time that paper was written we were not aware of the gap in the James/Whitehead classification so neither the theorem nor its proof can be found explicitly in [PetWil]. Because of the obvious importance of the result it seemed advisable to explain the details here, where in addition we will give a new proof.

The new proof is self-contained apart from two references to well-known results, and seems more natural. Unfortunately it does not yield an explicit diffeomorphism. The old proof gives an explicit diffeomorphism, but relies on references that are less well known, and it is harder to see where the argument is going. The new proof is given in section 1 and the details of the old proof are explained in section 2.

1. A FREE $S^3$-ACTION ON THE UNIT TANGENT BUNDLE

In this section we will give the new proof of Theorem 1.

First recall that the unit tangent bundle of $S^4$ is the bundle of type $(1,1)$. This is shown in Theorem 9.5 on page 99 of [Huse].

Next observe that the unit tangent bundle of $S^4$, $US^4$, admits a free $S^3$-action. To see this, view $S^4$ as a subset of $\mathbb{H} \times \mathbb{R}$ and let $S^3$ act on $\mathbb{H}$ by left quaternionic multiplication, and trivially on $\mathbb{R}$. The induced action on $S^4$ is not free. It fixes the points $(1,0)$ and $(0,-1)$, but is otherwise free. The differential of this action gives us the desired free $S^3$-action on $US^4$. To describe the action more explicitly, think of $US^4$ as

$$US^4 \equiv \{ (x,v) \in \mathbb{H} \times \mathbb{H} \mid |x| = 1, |v| = 1, (x,v) = 0 \}.$$ 

Our free $S^3$-action is then given by left multiplication in each factor.

What is the quotient of this action? Let $U_r$ be the set of vectors in the unit tangent bundle whose foot points lie in the metric sphere of radius $r$ about $(0,1)$ and let $\text{foot} : US^4 \rightarrow S^4$ be the foot point map. The action leaves the $U_r$'s invariant and for each $r$ in $(0,\pi)$ the quotient of $U_r$ is diffeomorphic to $S^3$. To see this fix $x$ in $\text{foot}(U_r)$ and observe that every vector whose foot point is $x$ is in a unique orbit and that every orbit in $U_r$ has a vector whose foot point is $x$. Thus $U_r/S^3$ is diffeomorphic to the set of vectors whose foot point is $x$, that is, to $S^3$.

On the other hand, the $S^3$-action is transitive on $U_0$ and $U_{\pi}$ so the quotient of each of these is a point. It follows that $US^4/S^3$ is a smooth manifold that is homeomorphic to the suspension of $S^3$, that is, to $S^4$. Moreover, the differential structure is the standard one. This follows from the main theorem of [Cerf] and

**Proposition 1.** $US^4/S^3$ is a twisted 4-sphere. That is, it is diffeomorphic to a smooth manifold obtained by gluing together two copies of the four disk with a diffeomorphism of $S^3$.

**Proof.** We will find embeddings $\iota_1, \iota_{-1} : D^4 \rightarrow US^4/S^3$ and $C : S^3 \times [-1,1] \rightarrow US^4/S^3$ so that

$$C(S^3 \times [-1,1]) \cap \iota_{-1}(D^4) = C(S^3 \times \{-1\}) = \iota_{-1}(\partial D^4),$$

where $\partial D^4$ is the boundary of $D^4$.

This completes the proof.


\[ C(S^3 \times [-1, 1]) \cap \iota_1(D^4) = C(S^3 \times \{1\}) = \iota_1(\partial D^4), \text{ and} \]
\[ \iota_{-1}(\partial D^4) \cap \iota_1(\partial D^4) = \emptyset. \]
From this it follows that \( US^4/S^3 \) is a twisted sphere. Our proof that \( US^4 \) is homeomorphic to \( S^4 \) gave us implicitly an embedding of \( S^3 \times (-2, 2) \to US^4/S^3 \) whose image is the complement of the two points \( U_0/S^3, U_\pi/S^3 \). \( C \) is obtained by restricting this embedding to \( S^3 \times [-1, 1] \), after reparameterizing the interval part appropriately.

Let \( g_r \) denote the restriction of the product metric on \( S^4 \times \mathbb{R}^5 \) to \( US^4 \), and let \( q : US^4 \to US^4/S^3 \) be the quotient map. Both \textit{foot} and \( q \) are Riemannian submersions with respect to \( g_r \).

To get \( \iota_1 \) choose \( w_0 \) in \( U_0 \) and let \( G_{w_0} \) be the set of \textit{q}-horizontal, normal geodesics emanating from \( w_0 \). Notice that \( G_{w_0} \) gives us a map \( e_{w_0} : D^4 \to US^4 \), defined via exponentiation, that is an embedding on a sufficiently small neighborhood of \( 0 \). Since the geodesics in \( G_{w_0} \) are \textit{q}-horizontal, \( q \circ e_{w_0} \) is also an embedding on a sufficiently small neighborhood of \( 0 \). Set \( \iota_1 \) equal to the restriction of \( q \circ e_{w_0} \) to this neighborhood. To see that the images of \( \iota_1 \) and \( C \) intersect in the appropriate way notice that \( U_0 \) is a fiber for \textit{foot} as well as for \( q \). Thus the geodesics in \( G_{w_0} \) are \textit{foot}-horizontal, and \( \textit{foot} \circ e_{w_0} \) is an embedding of a neighborhood of \( 0 \) that takes the ball of radius \( r \) about \( 0 \) to the ball of radius \( r \) about \((1, 0)\). From this it follows that the images of \( \iota_1 \) and \( C \) intersect in the desired manner, provided the "interval part" of \( C \) is adjusted in the appropriate way.

The map \( \iota_{-1} \) is defined in the same way as \( \iota_1 \) with the role of \((1, 0)\) being played by \((-1, 0)\). \( \square \)

We now know that in addition to being the bundle of type \((1, 1)\), the total space of the unit tangent bundle is a principal \( S^3 \)-bundle over \( S^4 \); that is, a bundle of type \((m, 0)\) or a bundle of type \((0, m)\). Among these bundles only four have the same homology as the bundle of type \((1, 1)\), namely the bundles of type \((2, 0)\), \((-2, 0)\), \((0, 2)\), and \((0, -2)\) ([PetWil], Proposition 8.2). Its not hard to see that the total spaces of these four principal bundles are mutually diffeomorphic, and hence that they are all diffeomorphic to the bundle of type \((1, 1)\). To see this note that the gluing maps satisfy \( g_{-2, 0} = g_{2, 1} \) and \( g_{0, -2} = g_{0, 1} \), so \( E_{2, 0} \cong E_{-2, 0} \) and \( E_{0, 2} \cong E_{0, -2} \). Finally notice that the map \( C : \mathbb{R}^4 \times S^3 \to \mathbb{R}^4 \times S^3 \) that is given by \( C(u, v) = (u, v) \) satisfies \( C \circ g_{2, 0} = g_{0, 2} \circ C \). So \( E_{2, 0} \cong E_{0, 2} \). Notice that the diffeomorphisms between the total spaces of the principal bundles take fibers to fibers.

Thus \( q : US^4 \to US^4/S^3 \) is one of the bundles \( E_{2, 0}, E_{-2, 0}, E_{0, 2}, \text{ or } E_{0, -2} \), and the identity map of \( US^4 \) is therefore a diffeomorphism from the total space of \( E_{1, 1} \) to the total space of \( q : US^4 \to US^4/S^3 \). Since \( U_0 \) is a fiber of both bundles, our diffeomorphism takes a fiber to a fiber.

2. An Explicit Diffeomorphism between \( E_{2, 0} \) and \( E_{1, 1} \)

Let \( Sp(2) \) denote the group of \( 2 \times 2 \)-unitary matrices with quaternion entries, that is, those that satisfy \( QQ^* = Q^*Q = id \), where \( Q^* \) is conjugate transpose. \( S^3 \) acts freely on \( Sp(2) \) via left multiplication by the matrices
\[
\begin{pmatrix}
q & 0 \\
0 & q
\end{pmatrix}, \quad q \in S^3.
\]
The quotient, $E_{2,0}$, admits two different submersions onto $S^4$, $p_{2,0}$ and $p_{1,1}$. It turns out that these are the $S^3$-bundles over $S^4$ of types $(2,0)$ and $(1,1)$ respectively. $p_{2,0}$ is the map given by

$$p_{2,0} : \text{orbit} \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \mapsto \tilde{h} \left( \begin{array}{c} b \\ d \end{array} \right),$$

where $\tilde{h} : S^7 \to S^4$ is the Hopf fibration given by left quaternionic multiplication. To see that $p_{2,0}$ is a principal $S^3$-bundle, just observe that the $S^3$-action on $Sp(2)$ given by right multiplication by

$$\left( \begin{array}{cc} q & 0 \\ 0 & 1 \end{array} \right)$$

commutes with the left action above, so they combine to give an $S^3 \times S^3$-action (cf. [GromMey]). The $S^3 \times S^3$-action is free, so the right action induces a free $S^3$-action on $E_{2,0} = Sp(2)/S^3$. This action on $E_{2,0}$ clearly leaves the fibers of $p_{2,0}$ invariant, so $E_{2,0}$ is a principal $S^3$-bundle over $S^4$. It was shown in [Rig, p. 192] that it is in fact the bundle of type $(2,0)$.

The map $p_{1,1}$ is given by

$$p_{1,1} : \text{orbit} \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \mapsto \tilde{h} \left( \begin{array}{cc} b & c \\ d & d \end{array} \right).$$

It was shown in Proposition 8.1 on page 362 of [PetWil] that this is the bundle of type $(1,1)$.

There is also a Lie Groups proof that $(E_{2,0}, p_{1,1})$ is $US^4$. To see this recall that $SO(5)$ acts transitively on $US^4$ by differentiating the $SO(5)$-action on $S^4$. The isotropy of a unit tangent vector is $SO(3)$, so $US^4 = SO(5)/SO(3)$. Note that the embedding of $SO(3)$ in $SO(5)$ is just the composition of the standard embeddings $SO(3) \hookrightarrow SO(4) \hookrightarrow SO(5)$. Taking the double covers of $SO(3)$ and $SO(5)$ we get that $US^4 = Spin(5)/Spin(3)$, and the embedding of $Spin(3)$ in $Spin(5)$ is standard. Now recall that $Sp(2)$ and $Spin(5)$ are isomorphic Lie groups, and also that $S^3$ and $Spin(3)$ are isomorphic. It follows that there is some embedding of $S^3$ in $Sp(2)$ so that $Sp(2)/S^3$ is the unit tangent bundle.

To see that this embedding is the one we have chosen view $S^7$ as the set of $2 \times 1$ quaternion matrices of unit length, and note that $Sp(2)$ acts by left multiplication on $S^7$. It was shown in Proposition 4.1 on page 183 of [GluWarZil] that this left action of $Sp(2)$ is a group of symmetries of the Hopf fibration $h : S^7 \to S^4$ that is given by right quaternionic multiplication. They also showed that the induced action on $S^4$ is the standard $\mathbb{Z}_2$-ineffective $Spin(5)$-action. (Also giving a very nice proof that $Sp(2)$ and $Spin(5)$ are isomorphic.)

Thus to check that our embedding of $S^3$ in $Sp(2)$ is the standard one, we just need to check that the action that $S^3$ induces on $S^4$ via $h$ is standard. This is easy to check using the explicit formula for $h$ in [Wil]. Namely

$$h \left( \begin{array}{c} a \\ c \end{array} \right) = \left( a\bar{c}, \frac{1}{2} \left( |a|^2 - |c|^2 \right) \right).$$
Thus the identity map of $\text{Sp}(2)/S^3$ is a diffeomorphism between the total spaces of $(\text{Sp}(2)/S^3, p_{2,0})$ and $(\text{Sp}(2)/S^3, p_{1,1})$. Since

$$p_{2,0}^{-1}(\bar{h}(0,1)) = \left\{ \text{orbit} \left( \begin{array}{cc} q_1 & 0 \\ 0 & q_2 \end{array} \right) \bigg| q_1, q_2 \in S^3 \right\} = p_{1,1}^{-1}(\bar{h}(0,1)),$$

we have constructed another diffeomorphism that takes a fiber to a fiber.

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REFERENCES


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